

Regularized One Dimensional Coulomb Potential Induced by the Presence of Minimal Lengths

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Abstract

In this paper we address the problem of a particle moving in singular one dimensional potentials in the framework of quantum mechanics with minimal length. Using the momentum space representation we solve exactly the Schrodinger equation for the Dirac delta potential and Coulomb potential. The effect of the minimal length is revealed by a computation of effective generalized Dirac delta potential and Coulomb potential.

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1 Introduction

In a serie of papers Kempf et al. [1, 2, 3] introduced a deformed quantum mecahnics based on generalized uncertainty principle (GUP). Other similar issues leading to the same GUP have been also initiated by some authors [4, 5, 6, 7]. One major consequence of the GUP is the appearance of minimal uncertainties in position and/or momentum leading to an UV/IR mixing, which allows to probe short distance physics (UV) from long distance one (IR). A minimal uncertainty in position or minimal length has appeared in different context like string theory [8, 9, 10], loop quantum gravity [11] and non-commutative field theories [12, 13]. The minimal length defines a scale below it the physics becomes inaccessible, leading to a natural cut-off which prevents from the usual UV divergencies.

Recently the Schrodinger equation in momentum space for the harmonic oscillator with minimal lengths in arbitrary dimensions has been solved [1, 2, 14]. The cosmological constant problem and the classical limit of the physics with minimal lengths have been also investigated [15, 16].

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On the other hand the effect of the minimal length on the properties of the 3D Coulomb potential has been studied in Refs. [17, 18, 19] and of the 3D Dirac oscillator using supersymmetric quantum mechanics [20]. The Casimir force for the electromagnetic field in the presence of the minimal length has been also computed [21, 22].

In this paper we are interested by the effect of the presence of minimal lengths on the energy eigenvalues and eigenfunctions of one dimensional quantum mechanical systems in singular potentials, like the attractive Dirac potential and Coulomb potential. In section 2, after implementing the minimal length via generalized position and momentum operators, we derive fundamental relations for our calculations. In sections 3 and 4, the one dimensional attractive Dirac potential and Coulomb potential are considered in great details. The last section is left for concluding remarks.

2 Quantum mechanics with minimal length

Following [7] we consider the following one dimensional realization of the position and momentum operators

$$X = i\hbar \exp(P^2/2\mu^2) \frac{d}{dp} \exp(P^2/2\mu^2) \quad P = p, \quad \beta \geq 0. \quad (1)$$

where μ is a parameter assumed to be large. The representation (1) leads to the following generalized commutator and generalized uncertainty principle (GUP)

$$[X, P] = i\hbar \exp(P^2/\mu^2), \quad \Delta X \Delta P \geq \frac{\hbar}{2} \left[1 + \frac{(\Delta p)^2}{\mu^2} \right]. \quad (2)$$

A consequence of this relation, is the appearance of a minimal length given by

$$(\Delta x)_{min} = \frac{\hbar}{\mu}, \quad (3)$$

which means a loss of localized states in the x -space since we cannot probe the coordinates space with a resolution less than the minimal length. In this situation we use the momentum space representation and maximally localized states to define a quasi position representation [2]. Then the existence of a one parameter family of position basis related to the minimal length allows us to write $X |x^{ML}\rangle = x |x^{ML}\rangle$ where the vectors $|x^{ML}\rangle$ represent maximal localisation states in the sense that $\langle X \rangle_{ML} = x$. They are normalized states unlike the ones of ordinary quantum mechanics. In the following we derive necessary relations for our calculation taking in mind that we must recover the usual quantum mechanics in the limit $\mu \rightarrow \infty$.

Using the maximally localized vectors we derive the following maximally localized wavefunction

$$\langle x^{ML} | p \rangle = N \exp \left(-\frac{ix}{2\hbar} \mu \sqrt{\pi} \operatorname{erf} \left(\frac{p}{\mu} \right) - \frac{p^2}{2\mu^2} \right). \quad (4)$$

These states are far from being the well known plane waves. However in the limit $\mu \rightarrow \infty$ and choosing the normalization constant N as $1/\sqrt{2\pi\hbar}$ we recover the usual plane waves of ordinary quantum mechanics.

On the other hand the states defined by Eq.(4) are physical ones since

$$\langle \frac{p^2}{2m} \rangle = \frac{1}{4\pi m\hbar} \int p^2 dp \exp\left(-\frac{p^2}{\mu^2}\right) = \frac{\mu^{\frac{3}{2}}}{16\sqrt{\pi}m\hbar} \quad (5)$$

Using the completeness relation $1 = \int_{-\infty}^{+\infty} dx |x^{ML}\rangle \langle x^{ML}|$ and (4) we obtain

$$\langle p' | p \rangle = \frac{2e^{-\frac{1}{2\mu^2}(p^2+p'^2)}}{\mu\sqrt{\pi}} \delta\left(\operatorname{erf}\left(\frac{p}{\mu}\right) - \operatorname{erf}\left(\frac{p'}{\mu}\right)\right). \quad (6)$$

With the aid of the relation $\delta f(\lambda) = \sum_i \frac{\delta(\lambda - \lambda_i)}{f'(\lambda_i)}$, where λ_i are the roots of $f(\lambda)$, we finally get

$$\langle p' | p \rangle = \delta(p - p'). \quad (7)$$

From this equation we have the usual completeness relation for the eigenstates $|p\rangle$

$$\int dp |p\rangle \langle p| = \mathbf{1}. \quad (8)$$

Let us finally show that the maximally localized states $|x^{ML}\rangle$, like the coherent states, do not form an orthogonal set. Indeed we have

$$\begin{aligned} \langle y^{ML} | x^{ML} \rangle &= \int dp \langle y^{ML} | p \rangle \langle p | x^{ML} \rangle \\ &= \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} dp e^{-\frac{p^2}{\mu^2}} \exp\left\{-\frac{i(y-x)}{2\hbar} \mu\sqrt{\pi} \operatorname{erf}\left(\frac{p}{\mu}\right)\right\} \end{aligned} \quad (9)$$

Using the variable q defined by $q = \frac{\mu\sqrt{\pi}}{2} \operatorname{erf}\left(\frac{p}{\mu}\right)$ we have

$$\begin{aligned} \langle y^{ML} | x^{ML} \rangle &= \frac{1}{2\pi\hbar} \int_{-\frac{\mu\sqrt{\pi}}{2}}^{+\frac{\mu\sqrt{\pi}}{2}} dq \exp\left\{-\frac{i(y-x)}{\hbar} q\right\} \\ &= \frac{1}{\pi(y-x)} \sin\left(\frac{\mu\sqrt{\pi}}{2\hbar}(y-x)\right). \end{aligned} \quad (10)$$

The right hand is a well behaved function unlike the Dirac distribution of ordinary quantum mechanics. It is clear that the limit $\mu \rightarrow \infty$ restores the usual normalization $\langle x^{ML} | y^{ML} \rangle = \delta(x - y)$.

Consider now the momentum representation of the one dimensional Schrödinger equation for a particle of mass m in the potential $V(x)$,

$$\left(\frac{P^2}{2m} + V(X)\right) |\Psi\rangle = E |\Psi\rangle. \quad (11)$$

Let us project on the momentum states $|p\rangle$ and inserting the closure relations for the maximally localized states and the momentum states respectively we have

$$\frac{p^2}{2m}\Psi(p) + \int_{-\infty}^{+\infty} dp' \int_{-\infty}^{+\infty} dx \langle p | V(X) | x^{ML} \rangle \langle x^{ML} | p' \rangle \langle p' | \Psi \rangle = E\Psi(p). \quad (12)$$

Then using the expression of the quasi-position eigenvectors we obtain the following equation

$$\frac{p^2}{2m}\Psi(p) + \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} dp' e^{-\frac{1}{2\mu^2}(p^2+p'^2)} \int_{-\infty}^{+\infty} dx e^{-\frac{i\lambda\mu}{2\hbar}\sqrt{\pi}(\text{erf}(\frac{p}{\mu^2}) - \text{erf}(\frac{p'}{\mu^2}))} V(x)\Psi(p') = E\Psi(p), \quad (13)$$

which can be written as

$$\frac{p^2}{2m}\Psi(p) + \int_{-\infty}^{+\infty} dp' e^{-\frac{1}{2\mu^2}(p^2+p'^2)} V(p, p')\Psi(p') = E\Psi(p), \quad (14)$$

where the potential $V(p, p')$ is the generalized Fourier transform of the potential $V(x)$

$$V(p, p') = \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} dx e^{-\frac{i\lambda\mu}{2\hbar}\sqrt{\pi}(\text{erf}(\frac{p}{\mu^2}) - \text{erf}(\frac{p'}{\mu^2}))} V(x). \quad (15)$$

3 One dimensional delta potential

Let us begin our investigation of the effect of minimal length on eigenfunctions and eigenenergies by considering the one dimensional Dirac delta potential $V(x) = -\kappa\delta(x)$. This potential is known to shares some common properties with the Coulomb potential in the limit of strong coupling. Using the generalized Fourier transform (15) we obtain the usual expression

$$V(p - p') = -\frac{\kappa}{2\pi\hbar}. \quad (16)$$

Substituting in Schrödinger equation (14) we have

$$\left(\frac{p^2}{2m} - E\right)\Psi(p) - \frac{\kappa}{2\pi\hbar} e^{-\frac{1}{2\mu^2}p^2} \int_{-\infty}^{+\infty} dp' e^{-\frac{1}{2\mu^2}p'^2} \Psi(p') = 0. \quad (17)$$

This equation can be written in the following integral form

$$\Psi(p) = \lambda \int_{-\infty}^{+\infty} dp' K(p, p')\Psi(p') \quad (18)$$

with $\lambda = 1$ and $K(p, p')$ a separable kernel given by

$$K(p, p') = f(p)g(p'), \quad (19)$$

where

$$f(p) = \frac{m\kappa}{\pi\hbar} \frac{e^{-\frac{1}{2\mu^2}p^2}}{(p^2 - 2mE)}, \quad g(p) = e^{-\frac{1}{2\mu^2}p'^2}. \quad (20)$$

Writing the wave function in the following form

$$\Psi(p) = cf(p) \quad \text{with} \quad c = \int_{-\infty}^{+\infty} dp' g(p') \Psi(p'), \quad (21)$$

we deduce the equation

$$\Psi(p) \left[1 - \int_{-\infty}^{+\infty} dp g(p) f(p) \right] = 0 \quad (22)$$

which admits the obvious solution

$$\int_{-\infty}^{+\infty} dp g(p) f(p) = 1. \quad (23)$$

This equation is the spectral condition from which we extract the possible energy levels. Using the expressions of $f(p)$ and $g(p)$ and performing the integral we obtain

$$\frac{m\kappa}{p_0\hbar} \left[1 - \operatorname{erf}\left(\frac{p_0}{\mu}\right) \right] e^{\frac{p_0^2}{\mu^2}} - 1 = 0, \quad (24)$$

where we set $2mE = -p_0^2$ for bound states. An asymptotic expansion in $\frac{1}{\mu}$ gives the following equation

$$\left(1 + \frac{2m\kappa}{\hbar\sqrt{\pi}\mu} \right) p_0 - \frac{m\kappa}{\hbar} = 0. \quad (25)$$

Using the fact that $\frac{1}{\mu}$ is a small parameter, the solution of (25) gives the following energy eigenvalue

$$E = -\frac{m\kappa^2}{2\hbar^2} \left(1 - \frac{2m\kappa}{\hbar\sqrt{\pi}\mu} \right)^2. \quad (26)$$

An asymptotic expansion in μ gives

$$E = -\frac{m\kappa^2}{2\hbar^2} \left(1 - 4\frac{m\kappa}{\hbar\sqrt{\pi}\mu} + 4\left(\frac{m\kappa}{\hbar\sqrt{\pi}\mu}\right)^2 \right). \quad (27)$$

Using the normalization condition $\int dp |\Psi(p)|^2 = 1$ and the the following formula [23]

$$\begin{aligned} \int_0^\infty y^{\nu-1} (y + \gamma)^{\lambda-1} e^{-\frac{\beta}{y}} dy &= \beta^{\frac{\nu-1}{2}} \gamma^{\frac{\nu-1}{2} + \lambda} \Gamma(1 - \nu - \lambda) e^{\frac{\beta}{2\gamma}} W_{\frac{\nu-1}{2} + \lambda, -\frac{\nu}{2}} \left(\frac{\beta}{\gamma} \right), \\ |\arg \gamma| &< \pi, \quad \operatorname{Re}(1 - \lambda) > \operatorname{Re} \nu > 0. \end{aligned} \quad (28)$$

to calculate the constant c , the normalized wave function in momentum space is finally given by

$$\Psi(p) = \left[\mu^{-1/2} p_0^{-5/2} \sqrt{\pi} W_{-\frac{3}{4}, -\frac{3}{4}} \left(\frac{p_0^2}{\mu^2} \right) \right]^{-1/2} \frac{e^{-\frac{p_0^2}{4\mu^2}} e^{-\frac{1}{2\mu^2} p^2}}{(p^2 + p_0^2)}. \quad (29)$$

The maximally localized wave function $\Phi(x)$ are obtained from the generalized inverse Fourier transform of the momentum space wave function given by

$$\Phi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dp \exp \left(\frac{ix}{2\hbar} \mu \sqrt{\pi} \operatorname{erf} \left(\frac{p}{\mu} \right) - \frac{p^2}{2\mu^2} \right) \Psi(p). \quad (30)$$

Thus using eq.(28) we obtain

$$\Phi(x) = \frac{1}{\sqrt{2\pi\hbar}} \left[\mu^{-1/2} p_0^{-5/2} \sqrt{\pi} W_{-\frac{3}{4}, -\frac{3}{4}} \left(\frac{p_0^2}{\mu^2} \right) \right]^{-1/2} e^{-\frac{p_0^2}{4\mu^2}} \int_{-\infty}^{+\infty} dp e^{-\frac{p^2}{\mu^2}} \frac{\exp \left(\frac{ix}{2\hbar} \mu \sqrt{\pi} \operatorname{erf} \left(\frac{p}{\mu} \right) \right)}{(p^2 + p_0^2)}. \quad (31)$$

Using the residus theorem we show that the second integral vanishes and we obtain

$$\Phi(x) = \left[2\hbar \mu^{-1/2} p_0^{-9/2} \pi^{-5/2} W_{-\frac{3}{4}, -\frac{3}{4}} \left(\frac{p_0^2}{\mu^2} \right) \right]^{-1/2} e^{\frac{3p_0^2}{4\mu^2}} \left[e^{-\frac{|x|}{2\hbar} \mu \sqrt{\pi} \operatorname{erfi} \left(\frac{p_0}{\mu} \right)} \right], \quad (32)$$

where $\operatorname{erfi}(x) = -i \operatorname{erf}(ix)$ is the imaginary error function.

Let us now define $\Phi(p) = e^{\frac{p^2}{2\mu^2}} \Psi(p)$ and rewrite Schrödinger equation with an effective potential, in the manner of the theory without the minimal length,

$$\left(\frac{p^2}{2m} - E \right) \Phi(p) + \int_{-\infty}^{+\infty} dp' V_{\text{eff}}(p, p') \Phi(p') = 0, \quad (33)$$

where

$$V_{\text{eff}}(p) = -\frac{\kappa}{2\pi\hbar} e^{-\frac{p^2}{\mu^2}}. \quad (34)$$

Using the inverse Fourier transform we obtain the following regularized Dirac potential in one dimension (see Figure 1.)

$$V_{\text{eff}}(x) = -\frac{\kappa \mu \sqrt{\pi}}{2\pi\hbar} e^{-\left(\frac{\mu x}{2\hbar} \right)^2}. \quad (35)$$

The results in the case without the minimal length are obtained by observing, that for large μ , we have [23]

$$W_{-\frac{3}{4}, -\frac{3}{4}} \left(\frac{p_0^2}{\mu^2} \right) \xrightarrow{\mu \rightarrow \infty} \left(\frac{\pi \mu}{4p_0} \right)^{\frac{1}{2}}, \quad (36)$$

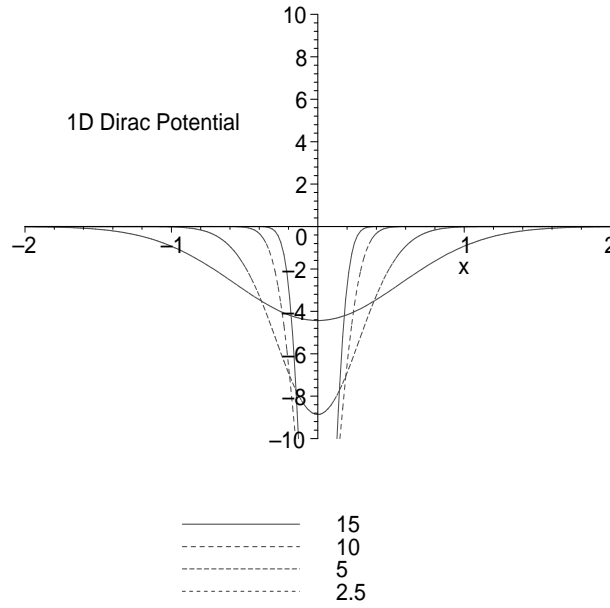


Figure 1: Effective one dimensional Dirac potential for different values of the minimal length.

and then the eigenfunctions $\Psi(p)$ and $\Phi(x)$ reduce respectively to the momentum and coordinate space eigenfunctions of the one dimensional Dirac potential without the minimal length.

$$\Psi(p) = \frac{\frac{m\kappa}{\hbar} \sqrt{\frac{2m\kappa}{\pi\hbar}}}{\left(p^2 + \left(\frac{m\kappa}{\hbar}\right)^2\right)}, \quad \Phi(x) = \sqrt{\frac{m\kappa}{\hbar^2}} e^{-\frac{m\kappa}{\hbar^2}|x|}. \quad (37)$$

4 One dimensional coulomb potential

Finally we consider the one dimensional coulomb potential $V(X) = -\frac{Ze^2}{X}$ in the presence of a minimal length. This potential without the minimal length, which seems to have applications in semiconductors or insulators, has been solved with different results [24, 25]. In the presence of the minimal length of Kempf et al. [2], this potential has been recently solved but with some incorrectness [26].

Using the following formula

$$\lim_{\epsilon \rightarrow 0} \int_{-\infty}^{+\infty} \frac{e^{-ipz}}{z \pm i\epsilon} dz = \mathbf{P} \int_{-\infty}^{+\infty} \frac{e^{-ipz}}{z} dz \mp i\pi, \quad (38)$$

where \mathbf{P} means the Cauchy principal value of the integral, the Coulomb potential in momentum

space is then given by

$$V(p, p') = \frac{iZe^2}{2\hbar} \left[\text{sign} \left(\text{erf} \left(p\sqrt{\beta} \right) - \text{erf} \left(p'\sqrt{\beta} \right) \right) + 1 \right]. \quad (39)$$

Substituting in Schrodinger's equation (14), we obtain

$$\left(\frac{p^2}{2m} - E \right) \Psi(p) + \frac{iZe^2}{\hbar} e^{-\frac{1}{2\mu^2}p^2} \left[\int_{-\infty}^p dp' e^{-\frac{1}{2\mu^2}p'^2} + \int_{-\infty}^{+\infty} dp' e^{-\frac{1}{2\mu^2}p'^2} \right] \Psi(p') = 0, \quad (40)$$

where the last term $\mathbf{S} = \int_{-\infty}^{+\infty} dp' e^{-\frac{1}{2\mu^2}p'^2} \Psi(p')$ is the signature of the singularity of Coulomb potential in one dimension in coordinates space.

Deriving (40) with respect to p gives

$$\Psi'(p) + \frac{2p}{(p^2 - 2mE)} \Psi(p) + \frac{p}{\mu^2} \Psi(p) + \frac{2miZe^2}{\hbar} \frac{e^{-\frac{1}{\mu^2}p^2}}{(p^2 - 2mE)} \Psi(p) = 0. \quad (41)$$

The solution to this equation is given by

$$\Psi_n(p) = N \frac{e^{-\frac{p^2}{2\mu^2}}}{(p^2 + p_0^2)} \exp \left(-\frac{2miZe^2}{\hbar} \int \frac{e^{-\frac{1}{\mu^2}p^2}}{(p^2 + p_0^2)} dp \right). \quad (42)$$

with the normalization constant, calculated in section 3, given by

$$N = \left[\mu^{-1/2} p_0^{-5/2} \sqrt{\pi} e^{\frac{p_0^2}{2\mu^2}} W_{-\frac{3}{4}, -\frac{3}{4}} \left(\frac{p_0^2}{\mu^2} \right) \right]^{-1/2}. \quad (43)$$

Expanding the last term in $1/\mu^2$ and integrating over p we obtain

$$\Psi_n(p) = N \frac{\exp \left(-\frac{p^2}{2\mu^2} + \frac{2miZe^2}{\hbar\mu^2} p \right)}{(p^2 + p_0^2)} \exp \left(-\frac{2miZe^2}{\hbar p_0} \left(1 + \frac{p_0^2}{\mu^2} \right) \arctan \frac{p}{p_0} \right). \quad (44)$$

The requirement that the wave function must be single valued gives the spectral condition

$$\left(\frac{p_0^2}{\mu^2} - \frac{\hbar n}{mZe^2} p_0 + 1 \right) = 0. \quad (45)$$

Solving this equation and using $E_n = -\frac{p_0^2}{2m}$ we obtain

$$E_n^\pm = -\frac{\mu^2}{8m} \left(\frac{\hbar n \mu}{mZe^2} \right)^2 \left(1 \pm \sqrt{1 - \left(\frac{2mZe^2}{\hbar n \mu} \right)^2} \right)^2 \quad (46)$$

An asymptotic expansion in μ gives

$$E_n^- = -\frac{mZ^2e^4}{2\hbar^2n^2} \left(1 - 2 \left(\frac{mZe^2}{\hbar n \mu} \right)^2 \right) \quad (47)$$

and

$$E_n^+ = -\frac{\mu^2}{2m} \left(-2 + \left(\frac{\hbar n \mu}{m Z e^2} \right)^2 - \left(\frac{m Z e^2}{\hbar n \mu} \right)^2 \right). \quad (48)$$

The first term in (47) is the spectrum of the usual one dimensional Coulomb potential [25], while the remaining ones are the corrections due to the perturbation of the space by the presence of the minimal length. The correction term in (47) is in concordance with the ones to the energy spectrum of Coulomb potential in three dimensions derived in [17, 18]. In [26] an additional correction term proportional to $\sqrt{\beta} \left(\frac{1}{\mu} \right)$ in the paper has been taken into account. However as noted in [18], this term like $e^{\frac{2miZe^2}{\hbar\mu^2}p}$ in eq.(21), will affect the outgoing scattered waves. The wave functions associated with the energy levels given by (48) are physically acceptable ones unless μ is large but finite. In the two cases we can extract a bound for the minimal length. In fact requiring that $E_n^\mp < 0$ for bound states we obtain

$$(\Delta x)_{\min} \leq \frac{\hbar^2 n}{\sqrt{2} Z e^2 m}. \quad (49)$$

Setting $n = 1$ and the fact that Bohr radius is 5.292×10^{-11} m we obtain

$$(\Delta x)_{\min} \leq 3.742 \times 10^{-11} \text{ m}. \quad (50)$$

Let us finally discuss about the original singularity of Coulomb potential in one dimension expressed by $\mathbf{S} = \int_{-\infty}^{+\infty} dp' e^{-\frac{1}{2\mu^2}p'^2} \Psi(p')$. As noted above, \mathbf{S} is an account of the discontinuity at the origin of the wave function in the coordinates space. With the aid of the wavefunctions given by (44) we show that

$$\begin{aligned} \mathbf{S} &= \lim_{p \rightarrow \infty} \frac{i\hbar e^{\frac{1}{2\mu^2}p^2}}{2mZe^2} (p^2 + p_0^2) \Psi_n(p) \\ &\rightarrow \lim_{p \rightarrow \infty} \exp\left(\frac{2miZe^2}{\hbar\mu^2}p\right). \end{aligned} \quad (51)$$

As it is obvious from (51), the singularity is completely removed by the presence of the minimal length. This is a natural consequence since the minimal length play the role of a cut-off which suppress the contribution of high momentum. This was not the situation in [26].

The regularizing effect of the minimal length is best seen by computing the effective Coulomb potential induced by the presence of the minimal. Indeed reappeting the same calculations leading to (35) along with $\mathbf{S} = 0$ and taking the inverse Fourier sine transform we obtain

$$V_{\text{eff}}(x) = -\frac{Ze^2\sqrt{\pi}}{2\hbar} \mu \text{erfi}\left(\frac{\mu x}{2}\right) e^{-\left(\frac{\mu x}{2\hbar}\right)^2}. \quad (52)$$

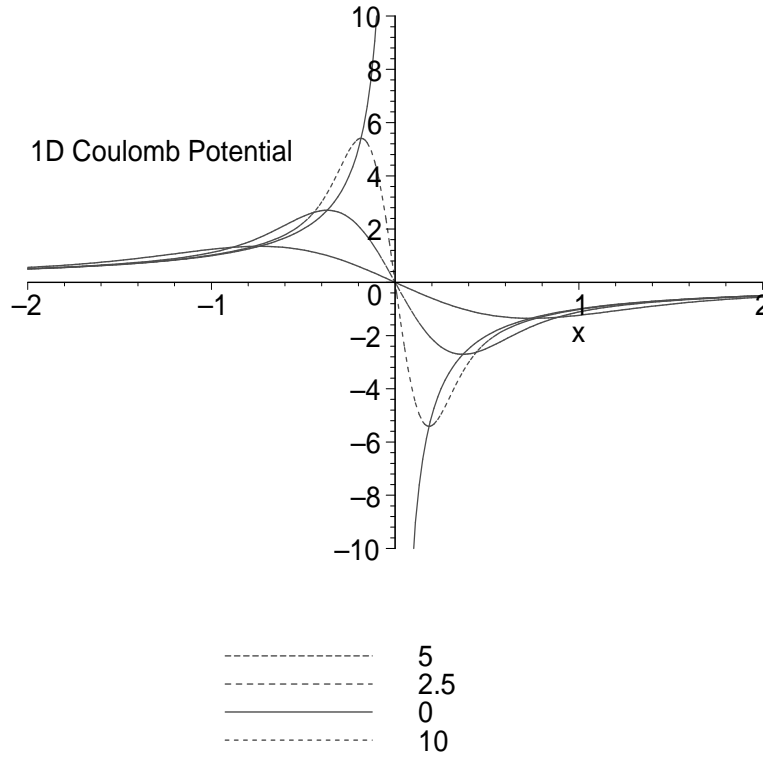


Figure 2: Figure 2: Effective one dimensional Coulomb potential for different values of the minimal length.

We note that the usual divergence is removed and the presence of the minimal length regularizes the theory (see Figure 2 below). Similar effective Coulomb potential in three dimensions induced by the non commutativity of the space-time coordinates have been recently computed [27]

5 Conclusion

In this paper we have solved, in the framework of quantum mechanics in the presence of minimal lengths, the problem of a particle moving in simple singular one dimensional potentials namely, the Dirac delta potential and the Coulomb potential. Our calculations are based on using physical maximal localisation states. In the two cases We have calculated the effective potential induced by the presence of the minimal length and as a consequence we have shown that the singularity at the origin, present in the case without the minimal length, is now completely removed. For a particle in the Coulomb potential the corrections to the energy spectrum are in concordance with that of Refs.[17, 18].

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